The blow-up and lifespan of solutions to systems of semilinear wave equation with critical exponents in high dimensions

Yuki Kurokawa [a] Hiroyuki Takamura [b] Kyouhei Wakasa [c]

[a] *General Education, Yonago National College of Technology, 4448 Yonago, Tottori 683-8502, Japan. e-mail : kurokawa@yonago-k.ac.jp*

[b] *Department of Complex and Intelligent Systems, Faculty of Systems Information Science, Future University Hakodate, 116-2 Kamedanakano-cho, Hakodate, Hokkaido 041-8655, Japan. e-mail : takamura@fun.ac.jp*

[c] *The 1st year of Graduate School of Systems Information Science, Future University Hakodate, 116-2 Kamedanakano-cho, Hakodate, Hokkaido 041-8655, Japan. e-mail : g2111045@fun.ac.jp*

Abstract

In this paper we prove the blow-up theorem in the critical case for weakly coupled systems of semilinear wave equations in high dimensions. The upper bound of the lifespan of the solution is precisely clarified.

Key words : blow-up, lifespan, critical exponent, system, semilinear wave equation
MSC2010 : 35L71, 35L52, 35B33, 35B44

1 Introduction

Let us consider the following systems of semilinear wave equations;

$$\left\{\begin{array}{l} u_{tt}-\Delta u=|v|^p \\ v_{tt}-\Delta v=|u|^q \end{array}\right. \qquad \text{in} \quad \mathbf{R}^n\times[0,\infty) \tag{1.1}$$

for $p,q>1$ and $n\geq 2$ with the data of the form;

$$\left\{\begin{array}{l} u(x,0)=\varepsilon f_1(x),\ u_t(x,0)=\varepsilon g_1(x) \\ v(x,0)=\varepsilon f_2(x),\ v_t(x,0)=\varepsilon g_2(x) \end{array}\right. \qquad \text{for} \quad x\in\mathbf{R}^n, \tag{1.2}$$

where $\varepsilon>0$ is a small parameter. We assume that $f_1,f_2,g_1,g_2\in C_0^\infty(\mathbf{R}^n)$ for the simplicity.

In order to describe the results on (1.1), we set

$$F(p,q,n)\equiv\max\left\{\frac{q+2+p^{-1}}{pq-1},\ \frac{p+2+q^{-1}}{pq-1}\right\}-\frac{n-1}{2}. \tag{1.3}$$

DelSanto, Georgiev and Mitidieri [2] first showed that the system (1.1) with (1.2) for $n\geq 2$ has a global in time solution for sufficiently small ε if $F(p,q,n)<0$, while a solution for some positive data blows up in finite time if $F(p,q,n)>0$. When the blow-up occurs, it is known that the maximal time $T(\varepsilon)$, so-called "lifespan", of the existence of solutions for arbitrarily fixed data can be estimated as

$$c\varepsilon^{-F(p,q,n)^{-1}}\leq T(\varepsilon)\leq C\varepsilon^{-F(p,q,n)^{-1}}, \tag{1.4}$$

where c and C are positive constants independent of ε. See [1, 6, 11, 12, 13].

When $F(p,q,n)=0$, the non-existence of global in time solutions, namely $T(\varepsilon)<\infty$, was shown by DelSanto and Mitidieri [3] for $n=3$. Moreover, a sharp estimate of the lifespan;

$$\begin{array}{rlll} \exp\left(c\varepsilon^{-\min\{p(pq-1),q(pq-1)\}}\right) \le T(\varepsilon) \le \exp\left(C\varepsilon^{-\min\{p(pq-1),q(pq-1)\}}\right) & \text{for} & p\neq q, \\ \exp\left(c\varepsilon^{-p(p-1)}\right) \le T(\varepsilon) \le \exp\left(C\varepsilon^{-p(p-1)}\right) & \text{for} & p=q, \end{array} \tag{1.5}$$

was obtained for $n=2,3$ by [1, 10, 13]. In high dimensions, $n\ge 4$, only the lower bounds of the lifespan are estimated. See [6, 11, 12]. The upper bounds of the lifespan have not been obtained, because we have similar technical difficulties to the corresponding problem for single equations;

$$\left\{\begin{array}{ll} u_{tt}-\Delta u=|u|^p & \text{in} \quad \mathbf{R}^n\times[0,\infty) \\ u(x,0)=\varepsilon f(x),\ u_t(x,0)=\varepsilon g(x) & \end{array}\right. \tag{1.6}$$

for $n\ge 4$ and $p=p_0(n)$. To show the blow-up result in this case was an open problem for a long period. See [4, 5, 7, 8, 9, 14, 15, 16, 17, 18, 21, 22], etc. for the other cases on (1.6). Here, $p_0(n)$ is a positive root of the quadratic equation $2+(n+1)p-(n-1)p^2=0$. Note that $F(p,p,n)=0$ is equivalent to $p=p_0(n)$.

Finally, Yordanov and Zhang [20] or Zhou [23] independently obtained a blow-up result of $T(\varepsilon)<\infty$ for this open case. Later, Takamura and Wakasa [19] have succeeded to get the optimal estimate of the lifespan by introducing a new iteration argument based on the method in [20]. Zhou Yi and Han Wei [24] have recently reproved the theorem in [19] along with the method in [23].

Our aim in this article is to clarify the lifespan of a solution to the system (1.1) for the critical case in high space dimensions by employing the argument in [19].

Theorem 1 *Let $n\ge 4$ and $F(p,q,n)=0$ with $1<p\le q$. Assume that*

$$\begin{array}{l} (f_1,g_1)\in H^1(\mathbf{R}^n)\cap L^q(\mathbf{R}^n)\times L^2(\mathbf{R}^n),\ (f_2,g_2)\in H^1(\mathbf{R}^n)\times L^2(\mathbf{R}^n), \\ f_1(x)=f_2(x)=g_1(x)=g_2(x)=0 \text{ for } |x|>R>0 \end{array} \tag{1.7}$$

and that f_1,f_2,g_1,g_2 are non-negative, especially that g_1 and g_2 do not vanish identically. Moreover, suppose that the problem (1.1) with (1.2) has a solution $(u,v)\in C([0,T(\varepsilon));H^1(\mathbf{R}^n)\cap L^q(\mathbf{R}^n)\times H^1(\mathbf{R}^n)$ with $(u_t,v_t)\in(C([0,T(\varepsilon));L^2(\mathbf{R}^n)))^2$ satisfying

$$\mathrm{supp}(u,v,u_t,v_t)\subset\{(x,t)\in\mathbf{R}^n\times[0,T(\varepsilon))\ ;\ |x|\le t+R\}. \tag{1.8}$$

Then, there exists a positive constant $\varepsilon_0=\varepsilon_0(f_1,f_2,g_1,g_2,p,q,n,R)$ such that $T(\varepsilon)$ has to satisfy

$$T(\varepsilon)\le\left\{\begin{array}{ll} \exp(C\varepsilon^{-p(pq-1)}) & \text{if} \quad p\neq q \\ \exp(C\varepsilon^{-p(p-1)}) & \text{if} \quad p=q, \end{array}\right. \tag{1.9}$$

for $0<\varepsilon\le\varepsilon_0$, where C is a positive constant independent of ε.

Remark 1.1 *When $n\ge 4$ and $F(p,q,n)=0$ with $1<p\le q$, we have that $p\le 2$. Because $p_0(4)=2$ and $p_0(n)$ is monotonously decreasing in n. Moreover, we can take $R\ge 1$ in the proof of Theorem 1 without loss of the generality.*

Remark 1.2 *The counter case, $1<q\le p$, can be obtained by symmetricity of (1.1). In fact, if one substitutes p with q and u with v, the upper bounds in (1.5) immediately follow from Theorem1.*

The proof is based on the argument in [19]. A blow-up property of a nonlinear system of the ordinary differential inequalities will be employed to show a blow-up of a solution to (1.1). The growing up of L^p norm of v is crucial to get the sharp estimate of the lifespan. In order to obtain the growth of the norm by iteration argument, we will employ the integral inequalities of L^p norm of v and L^q norm of u as a frame of the iteration, which are based on the method of Yordanov and Zhang [20] . Such an argument was introduced by Takamura and Wakasa [19].

This paper is organized as follows. In the next section, we shall show a blow-up property for nonlinear systems of ordinary differential inequalities. By making use of this, we prove Theorem 1 in Section 3. In Section 4, we complete the iteration argument.

2 Blow-up for systems of ODIs with a critical balance

As stated in Introduction, we shall show a blow-up theorem for ordinary differential inequalities with a critical balance in exponents.

Lemma 2.1 *Let $p, q > 1$, $a > 0$, $\alpha, \beta \geq 0$ and $\alpha + p\beta = a(pq-1) + 2(p+1)$. Suppose that $U, V \in C^2([0,T])$ satisfy*

$$U(t) \geq Kt^a \qquad \text{for} \quad t \geq T_0, \tag{2.1}$$

$$U''(t) \geq A(t+R)^{-\alpha}|V(t)|^p \qquad \text{for} \quad t \geq 0, \tag{2.2}$$

$$V''(t) \geq B(t+R)^{-\beta}|U(t)|^q \qquad \text{for} \quad t \geq 0, \tag{2.3}$$

$$U(0) \geq 0,\ U'(0) > 0,\ V(0) \geq 0,\ V'(0) > 0, \tag{2.4}$$

where all A, B, K, R, T_0 are positive constants with $T_0 \geq R$. Then, T must satisfy that $T \leq 2T_1$ provided $K \geq K_0$, where

$$K_0 = \left[\frac{a^{2p+2}2^{\alpha+p\beta+2p+1}(q+2)^{2p+2}(q+1)^p}{AB^p}\left(1 - \frac{1}{2^{a\delta}}\right)^{-2p-2}\right]^{\frac{1}{pq-1}} \tag{2.5}$$

and

$$T_1 = \max\left\{T_0,\ \frac{(2q+3)U(0)}{U'(0)}\right\} \tag{2.6}$$

with a positive constant $\delta \in (0, (pq-1)/(2p+2))$.

Proof. We shall prove this lemma by contradiction. Assume that $T > 2T_1$. We note that

$$U(t) \geq 0,\ U'(t) > 0,\ V(t) \geq 0 \text{ and } V'(t) > 0 \quad \text{for} \quad t \geq 0 \tag{2.7}$$

by (2.2), (2.3) and (2.4). Multiplying (2.3) by $U'(t)$ and integrating it over $[0,t]$, we have

$$\begin{aligned} &U'(t)V'(t) - U'(0)V'(0) - \int_0^t U''(s)V'(s)ds \\ &\geq B\int_0^t (s+R)^{-\beta}U(s)^qU'(s)ds \\ &\geq B(t+R)^{-\beta}\int_0^t U(s)^qU'(s)ds \\ &= \frac{B}{q+1}(t+R)^{-\beta}\{U(t)^{q+1} - U(0)^{q+1}\}. \end{aligned}$$

Then, (2.7) gives us

$$U'(t)V'(t) > \frac{B}{q+1}(t+R)^{-\beta}\{U(t)^{q+1} - U(0)^{q+1}\}.$$

Multiplying this inequality by $U'(t)$ again and integrating it over $[0,t]$, we have

$$\begin{aligned}
&U'(t)^2V(t) - U'(0)^2V(0) - 2\int_0^t U'(s)U''(s)V(s)ds \\
&> \frac{B}{q+1}\int_0^t (s+R)^{-\beta}\{U(s)^{q+1} - U(0)^{q+1}\}U'(s)ds \\
&\geq \frac{B}{q+1}(t+R)^{-\beta}\int_0^t \{U(s)^{q+1} - U(0)^{q+1}\}U'(s)ds \\
&= \frac{B}{q+1}(t+R)^{-\beta}\phi(t),
\end{aligned}$$

where we set

$$\phi(t) = \frac{U(t)^{q+2} - U(0)^{q+2}}{q+2} - U(0)^{q+1}\{U(t) - U(0)\} \geq 0 \qquad \text{for} \quad t \geq 0. \tag{2.8}$$

Thus, by (2.2) and (2.7), we have

$$V(t) > \frac{B}{q+1}(t+R)^{-\beta}\phi(t)U'(t)^{-2}.$$

Substituting this expression into (2.2), we obtain that

$$U''(t)U'(t)^{2p} > A\left(\frac{B}{q+1}\right)^p (t+R)^{-\alpha-p\beta}\phi(t)^p \qquad \text{for } t \geq 0.$$

Multiplying this inequality by $U'(t)$ and integrating it over $[0,t]$, we have

$$\begin{aligned}
&\frac{1}{2p+2}\{U'(t)^{2p+2} - U'(0)^{2p+2}\} \\
&> A\left(\frac{B}{q+1}\right)^p \int_0^t (s+R)^{-\alpha-p\beta}\phi(s)^pU'(s)ds \\
&\geq A\left(\frac{B}{q+1}\right)^p (t+R)^{-\alpha-p\beta}\int_0^t \phi(s)^p\phi'(s)\{\phi'(s)\}^{-1}U'(s)ds.
\end{aligned} \tag{2.9}$$

By (2.8), one can easily see that

$$\{\phi'(s)\}^{-1}U'(s) = \{U(s)^{q+1} - U(0)^{q+1}\}^{-1} \geq U(t)^{-q-1} \qquad \text{for} \quad 0 < s \leq t.$$

Then, s-integral in (2.9) is estimated from below as follows.

$$s\text{-integral} \geq U(t)^{-q-1}\int_0^t \phi(s)^p\phi'(s)ds = \frac{1}{p+1}U(t)^{-q-1}\phi(t)^{p+1}. \tag{2.10}$$

Moreover, by the monotonicity of $U(t)$, we have

$$\begin{aligned}
\phi(t) \quad &\geq \quad \frac{U(t)^{q+1}\{U(t) - U(0)\}}{q+2} - U(0)^{q+1}\{U(t) - U(0)\} \\
&= \quad \{U(t) - U(0)\}\left\{\frac{U(t)^{q+1}}{q+2} - U(0)^{q+1}\right\} \\
&\geq \quad \{U(t) - U(0)\}U(t)^q\left\{\frac{U(t)}{q+2} - U(0)\right\}.
\end{aligned}$$

It also follows from the monotonicity of $U'(t)$ that $U(t) - U(0) \geq U'(0)t$. Thus we have

$$U(t) - U(0) \geq \frac{U(t)}{q+2} - U(0) \geq \frac{U(t)}{2(q+2)} + \frac{U'(0)t + U(0)}{2(q+2)} - U(0) \geq \frac{U(t)}{2(q+2)}$$

provided $t \geq (2q+3)U(0)/U'(0)$. Hence we get

$$\phi(t) \geq \frac{1}{4(q+2)^2} U(t)^{q+2} \qquad \text{for} \quad t \geq \frac{(2q+3)U(0)}{U'(0)}. \tag{2.11}$$

Therefore, it follows from (2.9), (2.10) and (2.11) that

$$U'(t) > \left[\frac{2A}{\{4(q+2)^2\}^{p+1}} \left(\frac{B}{q+1} \right)^p \right]^{\frac{1}{2p+2}} (t+R)^{-\frac{\alpha+p\beta}{2p+2}} U(t)^{\frac{pq+2p+1}{2p+2}} \quad \text{for } t \geq \frac{(2q+3)U(0)}{U'(0)}.$$

By the definition of K_0 in (2.5), this inequality can be rewritten as

$$U'(t) > aK_0^{-\frac{pq-1}{2p+2}} \left(1 - \frac{1}{2^{a\delta}} \right)^{-1} t^{-\frac{\alpha+p\beta}{2p+2}} U(t)^{\frac{pq+2p+1}{2p+2}}$$

if $t \geq R$. Now, we assume that $t \geq T_1$, where T_1 is the one in (2.6). Multiplying the last inequality by $U(t)^{-1-\delta} > 0$ with a constant $\delta \in (0, (pq-1)/(2p+2))$ and replacing $U(t)$ in the right hand side by (2.1), we have

$$U(t)^{-1-\delta} U'(t) > a \left(\frac{K}{K_0} \right)^{\frac{pq-1}{2p+2}} K^{-\delta} \left(1 - \frac{1}{2^{a\delta}} \right)^{-1} t^{-1-a\delta}$$

because of the critical balance, $\alpha + p\beta = a(pq-1) + 2(p+1)$. Integrating the above inequality over $[T_1, t]$, we have

$$\frac{1}{\delta} \left\{ U(T_1)^{-\delta} - U(t)^{-\delta} \right\} > \frac{1}{\delta} \left(\frac{K}{K_0} \right)^{\frac{pq-1}{2p+2}} K^{-\delta} \left(1 - \frac{1}{2^{a\delta}} \right)^{-1} \left(T_1^{-a\delta} - t^{-a\delta} \right).$$

By the assumption of $T > 2T_1$, one can set $t = 2T_1$. Then, neglecting the second term in the left hand side, we get

$$1 > \left(\frac{K}{K_0} \right)^{\frac{pq-1}{2p+2}} \left\{ \frac{U(T_1)}{KT_1^a} \right\}^{\delta}.$$

By making use of (2.1) again for the right hand side, we find that this inequality implies

$$1 > \left(\frac{K}{K_0} \right)^{\frac{pq-1}{2p+2}}.$$

This contradicts to $K \geq K_0$. Therefore we have $T \leq 2T_1$. Lemma 2.1 is now established. □

3 Proof of Theorem 1

In this section we shall prove Theorem 1 by using Lemma 2.1. Let us define

$$U(t) = \int_{\mathbf{R}^n} u(x,t)dx \quad \text{and} \quad V(t) = \int_{\mathbf{R}^n} v(x,t)dx, \tag{3.1}$$

where (u, v) is a solution to (1.1) with (1.2) satisfying (1.8).

First, we shall show (2.2)-(2.4) in Lemma 2.1 for U and V in (3.1). Integrating two equations of (1.1) in $x \in \mathbf{R}^n$, we have

$$U''(t) = \int_{\mathbf{R}^n} |v(x,t)|^p \, dx \quad \text{and} \quad V''(t) = \int_{\mathbf{R}^n} |u(x,t)|^q \, dx$$

by the support property (1.8). By making use of Hölder's inequality together with the support property (1.8) again, we have

$$\begin{cases} U''(t) \geq B_n^{1-p}(t+R)^{-n(p-1)}|V(t)|^p \\ V''(t) \geq B_n^{1-q}(t+R)^{-n(q-1)}|U(t)|^q \end{cases} \quad \text{for} \quad t \geq 0,$$

where B_n stands for the volume of the unit ball in $\mathbf{R}^n$. This implies that (2.2) and (2.3) are valid with

$$\alpha = n(p-1),\ \beta = n(q-1),\ A = B_n^{1-p},\ B = B_n^{1-q}. \tag{3.2}$$

The assumption on the positiveness of data (1.7) gives us (2.4) with

$$\begin{aligned} U(0) &= \varepsilon \int_{\mathbf{R}^n} f_1(x)dx \geq 0, \qquad & U'(0) &= \varepsilon \int_{\mathbf{R}^n} g_1(x)dx > 0, \\ V(0) &= \varepsilon \int_{\mathbf{R}^n} f_2(x)dx \geq 0, \qquad & V'(0) &= \varepsilon \int_{\mathbf{R}^n} g_2(x)dx > 0. \end{aligned}$$

Next, we shall show the inequality (2.1) in this situation employing the following estimates of $U''(t)$ up to the case of $p \neq q$, or $p = q$.

Proposition 3.1 *Suppose that the assumptions in Theorem 1 are fulfilled and $p < q$. Then, $U(t)$ satisfies the following inequality.*

$$U''(t) \geq C_j (t - a_j R)^{n-1-(n-1)p/2} \left(\log \frac{t + (a_j^2 - 2)R}{(a_j+2)(a_j-1)R} \right)^{\frac{(pq)^j - 1}{pq-1}} \tag{3.3}$$

for $t \geq a_j R$ and $j = 1, 2, 3 \cdots$. Here we set $a_j = 6 \cdot 4^{j-1} - 2$ and

$$C_1 = \tilde{C}\varepsilon^{p^2 q}, \quad C_j = \exp\{(pq)^{j-1}(\log C_1 + S_j)\} \text{ for } j \geq 2, \tag{3.4}$$

where $\tilde{C} = \tilde{C}(p,q,n)$ is a positive constant and S_j is a convergent sequence independent of ε and t.

Proposition 3.2 *Suppose that the assumptions in Theorem 1 are fulfilled and $p = q$. Then, $U(t)$ satisfies the following inequality.*

$$U''(t) \geq D_j (t - b_j R)^{n-1-(n-1)p/2} \left(\log \frac{t + (b_j^2 - 6)R}{(b_j-2)(b_j+3)R} \right)^{\frac{p^{2j} - 1}{p-1}} \tag{3.5}$$

for $t \geq b_j R$ and $j = 1, 2, 3 \cdots$. Here we set $b_j = 10 \cdot 4^{j-1} - 2$ and

$$D_1 = \tilde{C}\varepsilon^{p^3}, \quad D_j = \exp\{p^{2(j-1)}(\log D_1 + S'_j)\} \text{ for } j \geq 2, \tag{3.6}$$

where $\tilde{C} = \tilde{C}(p,q,n)$ is a positive constant and S'_j is a convergent sequence independent of ε and t.

These propositions are proved in the next section. From now on, we shall apply (3.3) and (3.5) to the proof of (2.1). We are concentrated on the case $p<q$ only since the proof for the case of $p=q$ is analogue.

First, let $t \geq a_j R$. Integrating (3.3) over $[a_j R, t]$, we have

$$U'(t)-U'(a_jR) \geq C_j \int_{a_jR}^t (s-a_jR)^{n-1-(n-1)p/2} \left(\log \frac{s+(a_j^2-2)R}{(a_j+2)(a_j-1)R} \right)^{\frac{(pq)^j-1}{pq-1}} ds. \tag{3.7}$$

We can neglect the second term in the left hand side by (2.7). Restricting the time interval to $t \geq (a_j+1)R$, we can replace the lower limit by $a_j t/(a_j+1)$ because of

$$a_jR \leq \frac{a_j}{a_j+1} t < t \qquad \text{for} \quad t \geq (a_j+1)R.$$

Then it follows that

$$s-a_jR \geq \frac{a_j}{a_j+1}t - a_jR = \frac{a_j}{a_j+1}\{t-(a_j+1)R\} \geq \frac{t-(a_j+1)R}{2}$$

and

$$\frac{s+(a_j^2-2)R}{(a_j+2)(a_j-1)R} \geq \frac{\frac{a_j}{a_j+1}t+(a_j^2-2)R}{(a_j+2)(a_j-1)R} = \frac{a_jt+(a_j+1)(a_j^2-2)R}{(a_j+1)(a_j+2)(a_j-1)R} \geq \frac{t+(a_j+1)^2R}{(a_j+1)(a_j+2)R}$$

for $a_jt/(a_j+1) \leq s \leq t$. Thus, we get by (3.7) that

$$\begin{aligned} U'(t) &\geq C_j \left(\frac{t-(a_j+1)R}{2} \right)^{n-1-(n-1)p/2} \left(\log \frac{t+(a_j+1)^2R}{(a_j+1)(a_j+2)R} \right)^{\frac{(pq)^j-1}{pq-1}} \int_{a_jt/(a_j+1)}^t ds \\ &\geq \frac{C_j}{2^{n-(n-1)p/2}a_j} \{t-(a_j+1)R\}^{n-(n-1)p/2} \left(\log \frac{t+(a_j+1)^2R}{(a_j+1)(a_j+2)R} \right)^{\frac{(pq)^j-1}{pq-1}} \end{aligned}$$

for $t \geq (a_j+1)R$ because of $a_j = 6 \cdot 4^{j-1}-2 \geq 4$. Integrating the last inequality over $[(a_j+1)R, t]$ and treating it in the similar way as above, we have

$$U(t) \geq \frac{C_j}{2^{2n+1-(n-1)p}a_j^2} \{t-(a_j+2)R\}^{n+1-(n-1)p/2} \left(\log \frac{t+(a_j+1)(a_j+2)R}{(a_j+2)^2R} \right)^{\frac{(pq)^j-1}{pq-1}}$$

for $t \geq (a_j+2)R$.

Let us again restrict the time interval to $t \geq (a_j+2)^4R^2$. Then it holds that

$$\frac{t+(a_j+1)(a_j+2)R}{(a_j+2)^2R} \geq \sqrt{t} \times \frac{\sqrt{t}}{(a_j+2)^2R} \geq \sqrt{t}$$

and

$$t-(a_j+2)R = \frac{t}{2} + \frac{t-2(a_j+2)R}{2} \geq \frac{t}{2}.$$

It follows from these inequalities and $a_j = 6 \cdot 4^{j-1}-2 \leq 2 \cdot 4^j$ that

$$U(t) \geq K_j(t) t^{n+1-(n-1)p/2} \qquad \text{for} \quad t \geq (a_j+2)^4R^2, \tag{3.8}$$

where we set

$$K_j(t)=\frac{C_jA}{16^j}\left(\log\sqrt{t}\right)^{\frac{(pq)^j-1}{pq-1}},\quad A=2^{-3n-4+3(n-1)p/2}. \tag{3.9}$$

Now, we consider (3.9) for $t\in[(a_j+2)^4R^2,\ (a_{j+1}+2)^4R^2]$. Then, by the definition of C_j (3.4), K_j in (3.9) can be rewritten as

$$K_j(t)=\exp\left\{(pq)^{j-1}\log L_j(t)+\log A-j\log 16-\frac{1}{pq-1}\log(\log\sqrt{t})\right\},$$

where

$$L_j(t)=C_1e^{S_j}(\log\sqrt{t})^{\frac{pq}{pq-1}}.$$

Since S_j converges to a certain number, there exists a constant $S=S(p,q,n)$ such that $S_j\geq S$ for any $j=1,2,3,\cdots$. It follows from the definition of C_1, (3.4), that $L_j\geq e$ holds provided

$$\varepsilon^{p(pq-1)}\log t\geq E, \tag{3.10}$$

where $E=2(\tilde{C}^{-1}e^{1-S})^{(pq-1)/pq}$.

We assume (3.10). Then it follows from $t\leq(a_{j+1}+2)^4R^2$ that

$$K_j(t)\geq\exp\left\{(pq)^{j-1}+\log A-j\log 16-\frac{1}{pq-1}\log(\log((a_{j+1}+2)^2R))\right\}.$$

Hence we can see that $K_j(t)$ goes to infinity if j tends to infinity. Therefore, for K_0 defined in (2.5) with (3.2), $a=n+1-(n-1)p/2$ and a constant $\delta\in(0,(pq-1)/(2p+2))$, there exists an integer $J=J(f_1,f_2,g_1,g_2,n,p,q,R)$ such that

$$K_j(t)\geq K_0\qquad\text{for}\quad t\in[(a_j+2)^4R^2,(a_{j+1}+2)^4R^2],$$

as far as $j\geq J$. This implies that

$$U(t)\geq K_0t^{n+1-(n-1)p/2}\qquad\text{for}\quad t\geq(a_J+2)^4R^2,$$

provided (3.10) is valid.

Now, we are in a position to prove Theorem1 by making use of Lemma2.1. Set

$$T_0(\varepsilon)=\exp(E\varepsilon^{-p(pq-1)}), \tag{3.11}$$

where E is the one in (3.10). Then there exists a positive constant $\varepsilon_0=\varepsilon_0(f_1,f_2,g_1,g_2,p,q,n,R)$ such that

$$T_0(\varepsilon)\geq(a_J+2)^4R^2\quad\text{and}\quad 2\max\left\{T_0(\varepsilon),\ \frac{(2q+3)U(0)}{U'(0)}\right\}\leq\exp(2E\varepsilon^{-p(pq-1)}) \tag{3.12}$$

holds for $0<\varepsilon\leq\varepsilon_0$. As we see, (2.1) is now established for $t\geq T_0(\varepsilon)$ with this ε. We also obtain other inequalities in Lemma2.1 with (3.2) and $a=n+1-(n-1)p/2$. Note that the condition in Lemma2.1 $\alpha+p\beta=a(pq-1)+2(p+1)$ is equivalent to the critical relation $F(p,q,n)=0$. In this way, when $T(\varepsilon)>T_0(\varepsilon)$, (2.1) holds for $t\in[T_0(\varepsilon),T(\varepsilon))$. Hence Lemme2.1 and (3.12) show that

$$t\leq 2\max\left\{T_0(\varepsilon),\ \frac{(2q+3)U(0)}{U'(0)}\right\}\leq\exp(2E\varepsilon^{-p(pq-1)}).$$

Taking a supremum over $t\in[T_0(\varepsilon),T(\varepsilon))$, we get

$$T(\varepsilon)\leq\exp(2E\varepsilon^{-p(pq-1)})\qquad\text{for}\quad 0<\varepsilon\leq\varepsilon_0.$$

When $T(\varepsilon)\leq T_0(\varepsilon)$, (1.9) is trivial. Therefore the proof of Theorem 1 is ended.

4 Iteration argument

In this section, we will prove Proposition 3.1 and Proposition3.2 by iteration argument. The following integral expressions of U'' and V'' are the frame in our iteration.

Proposition 4.1 *Suppose that the assumptions in Theorem 1 are fulfilled. Then, $U(t)$ and $V(t)$ satisfy*

$$U''(t) \geq C \int_0^{t-R} \frac{\rho^{n-1-(n-1)p/2} d\rho}{(t-\rho+R)^{(n-1)p/2}} \left(\int_0^{(t-\rho-R)/2} V''(s) ds \right)^p, \tag{4.1}$$

$$V''(t) \geq \frac{C}{(t+R)^{(n-1)q/2}} \int_0^{t-R} \frac{\rho^{n-1} d\rho}{(t-\rho+R)^{(n-1)q/2}} \left(\int_0^{(t-\rho-R)/2} U''(s) ds \right)^q \tag{4.2}$$

for $t \geq R$, where C is a positive constant independent of ε and t.

Proof. Recall that $p \leq 2$. Then, these inequalities can be immediately obtained by (2.14) and (2.21) in Yordanov and Zhang [20]. Therefore we shall omit the proof.

The first step of the iteration is the following estimate.

Proposition 4.2 *Suppose that the assumptions in Theorem 1 are fulfilled. Then, there exists a positive constant $C = C(f_2, g_2, n, p, R)$ such that*

$$U''(t) \geq C\varepsilon^p (t+R)^{n-1-(n-1)p/2} \qquad \text{for} \quad t \geq 0. \tag{4.3}$$

Proof. This inequality can be proved by the same way as (2.5') in Yordanov and Zhang [20]. The key estimates, (2.4) and Lemma 2.2 in [20], are obatined by the first and second equations in (1.1).

Let us continue to prove Proposition3.1 by making use of the two propositions above. Substituting (4.3) into $U''(s)$ in the s-integral in (4.2), we have

$$V''(t) \geq \frac{C^{q+1}\varepsilon^{pq}}{(t+R)^{(n-1)q/2}} \int_0^{t-R} \frac{\rho^{n-1} d\rho}{(t-\rho+R)^{(n-1)q/2}} \left(\int_0^{(t-\rho-R)/2} (s+R)^{n-1-(n-1)p/2} ds \right)^q.$$

for $t \geq R$. Putting the upper limit of the s-integral into a part of the negative power of $s+R$, we have

$$\begin{aligned} s\text{-integral} \ &\geq \left(\frac{t-\rho+R}{2} \right)^{-(n-1)p/2} \int_0^{(t-\rho-R)/2} s^{n-1} ds \\ &\geq \frac{(t-\rho-R)^n}{2^n n (t-\rho+R)^{(n-1)p/2}}. \end{aligned}$$

Hence we get

$$\begin{aligned} V''(t) \ &\geq \frac{C^{q+1}\varepsilon^{pq}}{2^{nq} n^q (t+R)^{(n-1)q/2}} \int_0^{t-R} \frac{\rho^{n-1}(t-\rho-R)^{nq}}{(t-\rho+R)^{(n-1)q(p+1)/2}} d\rho \\ &\geq \frac{C^{q+1}\varepsilon^{pq}}{2^{nq} n^q (t+R)^{(n-1)q(p+2)/2}} \int_0^{t-R} \rho^{n-1}(t-\rho-R)^{nq} d\rho. \end{aligned}$$

Cutting the domain of the ρ-integral, we have

$$\rho\text{-integral} \geq \left(\frac{t-R}{2} \right)^{nq} \int_0^{(t-R)/2} \rho^{n-1} d\rho = \frac{(t-R)^{nq+n}}{2^{nq+n} n}.$$

Thus we get

$$V''(t) \geq \frac{C'(t-R)^{nq+n}}{(t+R)^{(n-1)q(p+2)/2}} \qquad \text{for} \quad t \geq R, \tag{4.4}$$

where $C' = C^{q+1}\varepsilon^{pq}2^{-2nq-n}n^{-q-1}$. Next we shall set $t \geq 3R$ and substitute (4.4) into $V''(s)$ in (4.1). Then, it follows that

$$\begin{aligned}
U''(t) &\geq CC'^p \int_0^{t-3R} \frac{\rho^{n-1-(n-1)p/2}d\rho}{(t-\rho+R)^{(n-1)p/2}} \left(\int_R^{(t-\rho-R)/2} \frac{(s-R)^{nq+n}}{(s+R)^{(n-1)q(p+2)/2}}ds\right)^p \\
&\geq CC'^p \int_0^{t-3R} \frac{\rho^{n-1-(n-1)p/2}d\rho}{(t-\rho+R)^{(n-1)p(pq+2q+1)/2}} \left(\int_R^{(t-\rho-R)/2} (s-R)^{nq+n}ds\right)^p \\
&= \frac{CC'^p}{(nq+n+1)^p 2^{npq+(n+1)p}} \int_0^{t-3R} \frac{\rho^{n-1-(n-1)p/2}(t-\rho-3R)^{npq+(n+1)p}}{(t-\rho+R)^{(n-1)p(pq+2q+1)/2}}d\rho
\end{aligned}$$

for $t \geq 3R$. Here we again restrict the time interval to $t \geq a_1 R = 4R$. Then, it follows from

$$5(t-\rho-3R) \geq t-\rho+R \quad \text{for} \quad \rho \leq t-4R$$

and

$$(n-1)p(pq+2q+1)/2 - npq - (n+1)p = 1 \quad \text{for} \quad F(p,q,n) = 0$$

that

$$\begin{aligned}
\rho\text{-integral} &\geq \frac{1}{5^{(n-1)p(pq+2q+1)/2}} \int_{(t-4R)/2}^{t-4R} \frac{\rho^{n-1-(n-1)p/2}}{t-\rho-3R}d\rho \\
&\geq \frac{1}{5^{(n-1)p(pq+2q+1)/2}}(t-4R)^{-(n-1)p/2}\left(\frac{t-4R}{2}\right)^{n-1} \int_{(t-4R)/2}^{t-4R} \frac{d\rho}{t-\rho-3R} \\
&= \frac{(t-4R)^{n-1-(n-1)p/2}}{5^{(n-1)p(pq+2q+1)/2}2^{n-1}} \log\frac{t-2R}{2R}.
\end{aligned}$$

Since

$$\frac{t-2R}{2R} = \frac{t+8t-18R}{18R} \geq \frac{t+32R-18R}{18R} = \frac{t+14R}{18R}$$

holds for $t \geq 4R$, we get

$$U''(t) \geq C_1(t-4R)^{n-1-(n-1)p/2}\log\frac{t+14R}{18R},$$

where

$$\begin{aligned}
C_1 &= \frac{CC'^p}{(nq+n+1)^p 2^{npq+(n+1)p+n-1}5^{(n-1)p(pq+2q+1)/2}} \\
&= \frac{C^{pq+p+1}\varepsilon^{p^2q}}{(nq+n+1)^p n^{p(q+1)} 2^{3npq+(2n+1)p+n-1}5^{(n-1)p(pq+2q+1)/2}}
\end{aligned}$$

Therefore (3.3) is true for $j=1$.

Next we shall show (3.3) by induction. Assume that (3.3) for $t \geq a_j R$ holds and C_j is unknown here except for $j=1$ but will be determined later on. When $t \geq (2a_j+1)R$, substituting (3.3) into (4.2), we have

$$\begin{aligned}
V''(t) &\geq \frac{CC_j^q}{(t+R)^{(n-1)q/2}} \int_0^{t-(2a_j+1)R} \frac{\rho^{n-1}d\rho}{(t-\rho+R)^{(n-1)q/2}} \\
&\quad \times \left\{\int_{a_jR}^{(t-\rho-R)/2} (s-a_jR)^{n-1-(n-1)p/2}\left(\log\frac{s+(a_j^2-2)R}{(a_j-1)(a_j+2)R}\right)^{\frac{(pq)^j-1}{pq-1}} ds\right\}^q \\
&\geq \frac{CC_j^q}{(t+R)^{(n-1)q/2}} \int_0^{t-(2a_j+1)R} \frac{\rho^{n-1}I_j(\rho,t)^q}{(t-\rho+R)^{(n-1)q(p+1)/2}}d\rho,
\end{aligned}$$

where

$$I_j(\rho,t)=\int_{a_jR}^{(t-\rho-R)/2}(s-a_jR)^{n-1}\left(\log\frac{s+(a_j^2-2)R}{(a_j-1)(a_j+2)R}\right)^{\frac{(pq)^j-1}{pq-1}}ds.$$

Here we restricted the time interval to $t\geq(2a_j+2)R$ and cut the domain of ρ-integral to be $[0,t-(2a_j+2)R]$. Then, it follows from $(2a_j+1)R\leq t-\rho-R$ and

$$a_jR\leq\frac{a_j}{2a_j+1}(t-\rho-R)<\frac{t-\rho-R}{2}$$

that one can cut the domain of s-integral as $[a_j(t-\rho-R)/(2a_j+1),(t-\rho-R)/2]$. In this interval, we have

$$\begin{aligned}\frac{s+(a_j^2-2)R}{(a_j-1)(a_j+2)R}&\geq\frac{\frac{a_j}{2a_j+1}(t-\rho-R)+(a_j^2-2)R}{(a_j-1)(a_j+2)R}\\&=\frac{a_j(t-\rho-R)+(2a_j+1)(a_j^2-2)R}{(2a_j+1)(a_j-1)(a_j+2)R}\\&\geq\frac{(a_j-1)(t-\rho-R)+(2a_j+1)R+(2a_j+1)(a_j^2-2)R}{(2a_j+1)(a_j-1)(a_j+2)R}\\&=\frac{t-\rho+a_j(2a_j+3)R}{(2a_j+1)(a_j+2)R}\end{aligned}$$

and

$$s-a_jR\geq\frac{a_j}{2a_j+1}(t-\rho-R)-a_jR\geq\frac{t-\rho-(2a_j+2)R}{3}.$$

Hence we have

$$\begin{aligned}I_j(\rho,t)&\geq\left(\frac{t-\rho-(2a_j+2)R}{3}\right)^{n-1}\left(\log\frac{t-\rho+a_j(2a_j+3)R}{(2a_j+1)(a_j+2)R}\right)^{\frac{(pq)^j-1}{pq-1}}\\&\quad\times\left(\frac{1}{2}-\frac{a_j}{2a_j+1}\right)(t-\rho-R)\\&\geq\frac{1}{2\cdot3^na_j}(t-\rho-(2a_j+2)R)^n\left(\log\frac{t-\rho+a_j(2a_j+3)R}{(2a_j+1)(a_j+2)R}\right)^{\frac{(pq)^j-1}{pq-1}}\end{aligned}$$

because of $a_j \geq 1$ for any $j \in \mathbf{N}$. Therefore we obtain

$$
\begin{aligned}
V''(t) \geq & \frac{CC_j^q}{2^q \cdot 3^{nq} a_j^q (t+R)^{(n-1)q/2}} \int_0^{t-(2a_j+2)R} \frac{\rho^{n-1}\{t-\rho-(2a_j+2)R\}^{nq}}{(t-\rho+R)^{(n-1)q(p+1)/2}} \\
& \times \left(\log \frac{t-\rho+a_j(2a_j+3)R}{(2a_j+1)(a_j+2)R} \right)^{\frac{q((pq)^j-1)}{pq-1}} d\rho \\
\geq & \frac{CC_j^q}{2^q \cdot 3^{nq} a_j^q (t+R)^{(n-1)q(p+2)/2}} \\
& \times \int_0^{(t-(2a_j+2)R)/2} \rho^{n-1}\{t-\rho-(2a_j+2)R\}^{nq} \left(\log \frac{t-\rho+a_j(2a_j+3)R}{(2a_j+1)(a_j+2)R} \right)^{\frac{q((pq)^j-1)}{pq-1}} d\rho \\
\geq & \frac{CC_j^q}{2^{(n+1)q} \cdot 3^{nq} a_j^q} \cdot \frac{\{t-(2a_j+2)R\}^{nq}}{(t+R)^{(n-1)q(p+2)/2}} \left(\log \frac{t+(4a_j^2+8a_j+2)R}{2(2a_j+1)(a_j+2)R} \right)^{\frac{q((pq)^j-1)}{pq-1}} \\
& \times \int_0^{(t-(2a_j+2)R)/2} \rho^{n-1} d\rho \\
= & \frac{C_j'\{t-(2a_j+2)R\}^{nq+n}}{(t+R)^{(n-1)q(p+2)/2}} \left(\log \frac{t+(4a_j^2+8a_j+2)R}{2(2a_j+1)(a_j+2)R} \right)^{\frac{q((pq)^j-1)}{pq-1}},
\end{aligned}
$$

where

$$
C_j' = \frac{CC_j^q}{n 2^{(n+1)q+n} \cdot 3^{nq} a_j^q}. \tag{4.5}
$$

When $t \geq (4a_j+5)R$, replacing the $V''(s)$ in the right hand side in (4.1) by the last inequality above, we have

$$
\begin{aligned}
U''(t) \geq & CC_j'^p \int_0^{t-(4a_j+5)R} \frac{\rho^{n-1-(n-1)p/2}}{(t-\rho+R)^{(n-1)p/2}} d\rho \\
& \times \left(\int_{(2a_j+2)R}^{(t-\rho-R)/2} \frac{\{s-(2a_j+2)R\}^{nq+n}}{(s+R)^{(n-1)q(p+2)/2}} \left(\log \frac{s+(4a_j^2+8a_j+2)R}{2(2a_j+1)(a_j+2)R} \right)^{\frac{q((pq)^j-1)}{pq-1}} ds \right)^p \\
\geq & CC_j'^p \int_0^{t-(4a_j+5)R} \frac{\rho^{n-1-(n-1)p/2} J_j(\rho,t)^p}{(t-\rho+R)^{(n-1)p(pq+2q+1)/2}} d\rho,
\end{aligned}
$$

where

$$
J_j(\rho,t) = \int_{(2a_j+2)R}^{(t-\rho-R)/2} \{s-(2a_j+2)R\}^{nq+n} \left(\log \frac{s+(4a_j^2+8a_j+2)R}{2(2a_j+1)(a_j+2)R} \right)^{\frac{q((pq)^j-1)}{pq-1}} ds
$$

Let us again restrict the time interval to $t \geq (4a_j+6)R$ and cut the domain of ρ-integral to be $[0, t-(4a_j+6)R]$. Then, it follows from $(4a_j+5)R \leq t-\rho-R$ and

$$
(2a_j+2)R \leq \frac{2a_j+2}{4a_j+5}(t-\rho-R) \leq \frac{t-\rho-R}{2}
$$

that one can cut the domain of the s-integral of $J_j(\rho,t)$ to be $[(2a_j+2)(t-\rho-R)/(4a_j+5),(t-\rho-R)/2]$. Then, the inequality

$$
\begin{aligned}
&\frac{s+(4a_j^2+8a_j+2)R}{2(2a_j+1)(a_j+2)R} \geq \frac{\frac{2a_j+2}{4a_j+5}(t-\rho-R)+(4a_j^2+8a_j+2)R}{2(2a_j+1)(a_j+2)R}\\
&= \frac{(2a_j+2)(t-\rho-R)+(4a_j+5)(4a_j^2+8a_j+2)R}{2(4a_j+5)(2a_j+1)(a_j+2)R}\\
&\geq \frac{(2a_j+1)(t-\rho-R)+(4a_j+5)R+(4a_j+5)(4a_j^2+8a_j+2)R}{2(4a_j+5)(2a_j+1)(a_j+2)R}\\
&= \frac{t-\rho+(8a_j^2+22a_j+14)R}{2(4a_j+5)(a_j+2)R}
\end{aligned}
$$

holds for $(2a_j+2)(t-\rho-R)/(4a_j+5) \leq s \leq (t-\rho-R)/2$. Thus we have

$$
\begin{aligned}
J_j(\rho,t) &\geq \left(\log\frac{t-\rho+(8a_j^2+22a_j+14)R}{2(4a_j+5)(a_j+2)R}\right)^{\frac{q((pq)^j-1)}{pq-1}}\\
&\quad\times\int_{\frac{2a_j+2}{4a_j+5}(t-\rho-R)}^{(t-\rho-R)/2}\left\{s-\frac{2a_j+2}{4a_j+5}(t-\rho-R)\right\}^{nq+n}ds\\
&\geq \frac{(t-\rho-R)^{nq+n+1}}{(nq+n+1)(12a_j)^{nq+n+1}}\left(\log\frac{t-\rho+(8a_j^2+22a_j+14)R}{2(4a_j+5)(a_j+2)R}\right)^{\frac{q((pq)^j-1)}{pq-1}}
\end{aligned}
$$

because of

$$
\frac{t-\rho-R}{2}-\frac{2a_j+2}{4a_j+5}(t-\rho-R)=\frac{1}{2(4a_j+5)}(t-\rho-R)\geq\frac{t-\rho-R}{12a_j}.
$$

Hence we obtain

$$
\begin{aligned}
U''(t) &\geq \frac{CC_j'^p}{(nq+n+1)^p(12a_j)^{npq+(n+1)p}}\\
&\quad\times\int_0^{t-(4a_j+6)R}\frac{\rho^{n-1-(n-1)p/2}(t-\rho-R)^{npq+(n+1)p}}{(t-\rho+R)^{(n-1)p(pq+2q+1)/2}}d\rho\\
&\qquad\qquad\times\left(\log\frac{t-\rho+(8a_j^2+22a_j+14)R}{2(4a_j+5)(a_j+2)R}\right)^{\frac{pq((pq)^j-1)}{pq-1}}\\
&\geq \frac{CC_j'^p\{t-(4a_j+6)R\}^{-(n-1)p/2}}{(nq+n+q)^p(12a_j)^{npq+(n+1)p}}\\
&\quad\times\int_0^{t-(4a_j+6)R}\frac{\rho^{n-1}\{t-\rho-(4a_j+5)R\}^{npq+(n+1)p}}{(t-\rho+R)^{(n-1)p(pq+2q+1)/2}}d\rho\\
&\qquad\qquad\times\left(\log\frac{t-\rho+(8a_j^2+22a_j+14)R}{2(4a_j+5)(a_j+2)R}\right)^{\frac{pq((pq)^j-1)}{pq-1}}.
\end{aligned}
$$

Since

$$
t-\rho+R\leq(4a_j+7)\{t-\rho-(4a_j+5)R\}\leq 6a_j\{t-\rho-(4a_j+5)R\}
$$

is valid for $t-\rho\geq(4a_j+6)R$ and

$$
\frac{n-1}{2}p(pq+2q+1)-npq-(n+1)p=1
$$

holds for $F(p,q,n)=0$, the ρ-integral is dominated from below by

$$
\begin{aligned}
& \frac{\{t-(4a_j+6)R\}^{n-1}}{2^{n-1}(6a_j)^{(n-1)p(pq+2q+1)/2}} \\
& \quad \times \int_{(t-(4a_j+6)R)/2}^{t-(4a_j+6)R} \frac{d\rho}{t-\rho-(4a_j+5)R} \left(\log \frac{t-\rho+(8a_j^2+22a_j+14)R}{2(4a_j+5)(a_j+2)R} \right)^{\frac{pq((pq)^j-1)}{pq-1}} \\
& \geq \frac{\{t-(4a_j+6)R\}^{n-1}}{2^{n-1}(6a_j)^{(n-1)p(pq+2q+1)/2}} \\
& \quad \times \int_{(t-(4a_j+6)R)/2}^{t-(4a_j+6)R} \frac{d\rho}{t-\rho+(8a_j^2+22a_j+14)R} \left(\log \frac{t-\rho+(8a_j^2+22a_j+14)R}{2(4a_j+5)(a_j+2)R} \right)^{\frac{pq((pq)^j-1)}{pq-1}} \\
& \geq \frac{(pq-1)\{t-(4a_j+6)R\}^{n-1}}{2^{n-1}(6a_j)^{(n-1)p(pq+2q+1)/2}(pq)^{j+1}} \left(\log \frac{t+(16a_j^2+48a_j+34)R}{4(4a_j+5)(a_j+2)R} \right)^{\frac{(pq)^{j+1}-1}{pq-1}}.
\end{aligned}
$$

Setting $a_{j+1}=4a_j+6$, we get the desired inequality for $j+1$;

$$
U''(t) \geq C_{j+1}(t-a_{j+1}R)^{n-1-(n-1)p/2} \left(\log \frac{t+(a_{j+1}^2-2)R}{(a_{j+1}-1)(a_{j+1}+2)R} \right)^{\frac{(pq)^{j+1}-1}{pq-1}},
$$

where we set

$$
C_{j+1} = \frac{(pq-1)CC_j'^p}{(nq+n+1)^p 2^{npq+(n+1)p+n-1}(6a_j)^{(n-1)p^2q/2+(2n-1)pq+(3n+1)p/2}(pq)^{j+1}}.
$$

To end the proof, we shall fix all the coefficients, C_j. It follows from $a_j = 6\cdot 4^{j-1}-2 \leq 2\cdot 4^j$ and the definition of C_j', (4.5), that

$$
C_{j+1} = \frac{MC_j^{pq}}{N^j},
$$

where

$$
\begin{aligned}
M &= \frac{(pq-1)C^{p+1}}{pqn^p(nq+n+1)^p 2^{(n-1)p^2q+6npq+(5n+2)p+n-1} 3^{(n-1)p^2q/2+(3n-1)pq+(3n+1)p/2}}, \\
N &= 4^{(n-1)p^2q/2+2npq+(3n+1)p/2} pq.
\end{aligned}
$$

This equality is rewritten as

$$
\log C_{j+1} = pq \log C_j + \log M - j \log N.
$$

Then, one can easily get

$$
\begin{aligned}
\log C_{j+1} &= (pq)^j \log C_1 + \sum_{k=1}^{j} (pq)^{j-k} \log M - \sum_{k=1}^{j} k (pq)^{j-k} \log N \\
&= (pq)^j (\log C_1 + S_{j+1}),
\end{aligned}
$$

where we set

$$
S_j = \sum_{k=1}^{j-1} \frac{\log M - k \log N}{(pq)^k}. \tag{4.6}
$$

Note that S_j converges as $j \to \infty$. Therefore this completes the proof of Proposition 3.1.

We omit to show the proof of Proposition 3.2 because it is almost the same as the single case in Takamura and Wakasa [19]. The difference from the proof Proposition 3.1 appears in handling of logarithmic terms. In order to prove Proposition 3.2, we should integrate the logarithmic term at every steps in the iteration while such an integration is required only to get the estimate for $U''(t)$ in the proof of Proposition 3.1.

Acknowledgment

This manuscript was partially prepared during the first author's stay at University of Pisa in Italy from 4/4/2011 to 28/9/2011 as an overseas research fellow sponsored by Institute of National College of Technology, Japan. She is deeply grateful to Professor Vladimir Georgiev for his hearty hospitality, a lot of help on her stay and many fruitful discussions. She also thanks to all the members of Department of Mathematics, University of Pisa for preparations of the necessaries for her activities.

References

[1] R.Agemi, Y.Kurokawa and H.Takamura, *Critical curve for p-q systems of nonlinear wave equations in three space dimensions,* J.Differential Equations **167** (2000), 87-133.

[2] D.Del Santo, V.Georgiev and E.Mitidieri, *Global existence of the solutions and formation of singularities for a class of hyperbolic systems,* in "Geometric Optics and Related Topics" (F.Colombini and N.Lerner Eds.), Progress in Nonlinear Differential Equations and Their Applications, **32**, pp.117-140, Birkhäuser Boston, 1997.

[3] D.Del Santo and E.Mitidieri, *Blow-up of solutions of a hyperbolic system: the critical case,* Differential Equations, **34**, (1998), 1157-1163.

[4] S.Di Pomponio and V.Georgiev, *Life-span of subcritical semilinear wave equation,* Asymptotic Anal., 28(2001), 91-114.

[5] V.Georgiev, H.Lindblad and C.Sogge, *Weighted Strichartz estimates and global existence for semilinear wave equations,* Amer. J. Math.,**119** (1997), 1291-1319.

[6] V.Georgiev, H.Takamura and Zhou Yi, *The lifespan of solutions to nonlinear systems of a high-dimensional wave equation,* Nonlinear Anal. **64** (2006), no.10, 2215–2250.

[7] R.Glassey, *Finite-time blow-up for solutions of nonlinear wave equations,* Math. Z.,**177** (1981), 323-340.

[8] R.Glassey, *Existence in the large for* $\Box u = F(u)$ *in two space dimensions,* Math. Z.,**178** (1981), 233–261.

[9] F.John, *Blow-up of solutions of nonlinear wave equations in three space dimensions,* Manuscripta Math., **28** (1979), 235–268.

[10] H.Kubo and M.Ohta, *Critical blowup for systems of semilinear wave equations in low space dimensions,* J. Math. Anal. Appl., **240**, (1999), 340-360.

[11] Y.Kurokawa, *The lifespan of radially symmetric solutions to nonlinear systems of odd dimensional wave equations,* Nonlinear Anal. **60** (2005), no.7, 1239–1275.

[12] Y.Kurokawa, *The sharp lifespan of radially symmetric solutions to nonlinear wave systems in even space dimensions,* Mathematical Research Note, Univ. of Tsukuba #2004-001.

[13] Y.Kurokawa and H.Takamura, *A weighted pointwise estimate for two dimensional wave equations and its applications to nonlinear systems,* Tsukuba J. Math. **27** (2003), no.2, 417–448.

[14] Li Ta-Tsien and Zhou Yi, *A note on the life-span of classical solutions to nonlinear wave equations in four space dimensions,* Indiana Univ. Math. J., **44** (1995), 1207–1248.

[15] H.Lindblad, *Blow-up for solutions of $\Box u = |u|^p$ with small initial data,* Comm. Partial Differential Equations,**15(6)** (1990), 757-821.

[16] J.Schaeffer, *The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p,* Proc. Roy. Soc. Edinburgh, **101A** (1985), 31-44.

[17] T.C.Sideris, *Nonexistence of global solutions to semilinear wave equations in high dimensions,* J. Differential Equations, **52** (1984), 378-406.

[18] W.A.Strauss, Nonlinear Wave Equations, Conference Board of the Mathematical Sciences, Reginal Conference Series in Mathematics 73, AMS, 1989.

[19] H.Takamura and K.Wakasa, *The sharp upper bound of the lifespan of solutions to critical semilinear wave equations in high simensions,* J. Differential Equations **251** (2011), 1157-1171.

[20] B.Yordanov and Q.S.Zhang, *Finite time blow up for critical wave equations in high dimensions,* J. Funct. Anal. **231** (2006), no. 2, 361–374.

[21] Zhou Yi, *Blow up of classical solutions to $\Box u = |u|^{1+\alpha}$ in three space dimensions,* J. Partial Differential Equations, **5** (1992), 21–32.

[22] Zhou Yi, *Life span of classical solutions to $\Box u = |u|^p$ in two space dimensions,* Chin. Ann. of Math., **14B** (1993), 225–236.

[23] Zhou Yi, *Blow up of solutions to semilinear wave equations with critical exponent in high dimensions,* Chin. Ann. Math. Ser. B 28 (2007), no. 2, 205–212.

[24] Yi Zhou and Wei Han, *Life-span of solutions to critical semilinear wave equations,* 19 March 2011, arXiv:1103.3758 [math.AP].